

# STATISTICAL DAMAGE IDENTIFICATION TECHNIQUES APPLIED TO THE I-40 BRIDGE OVER THE RIO GRANDE RIVER

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## ABSTRACT

The statistical significance of vibration-based damage identification parameters is studied via application to the data from the tests performed on the Interstate 40 highway bridge in Albuquerque, New Mexico. A test of statistical significance is applied to the mean and confidence interval estimates of the modal properties and the corresponding damage indicators. The damage indicator used in this study is the change in the measured flexibility matrix. Previously presented deterministic results indicate that damage is detectable in all of the damage cases from these data sets. The results of this study indicate that the changes in both the modal properties and the damage indicators are statistically significant for all of the damage cases. However, these changes are distributed spatially for the first three damage cases and do not localize the damage until the fourth and final damage case.

## INTRODUCTION

Damage identification using changes in measured modal parameters is a topic that has received considerable attention in the literature in recent years. A review of the state of the art in the field is presented in Ref. [1]. The damage identification experiments on the I-40 bridge over the Rio Grande River in Albuquerque, New Mexico, described in Ref. [2], produced what is to date one of the most studied data sets in the field of vibration-based damage identification. A number of different researchers have analyzed these damage cases, including: Stubbs, et al. [3], Jauregui and Farrar [4], [5], [6], Alvin [7], Simmermacher, et al. [8], James, et al. [9], and Mayes [10].

However, one aspect of these data sets that has not been examined is the statistical significance of the damage identification results. Characterization of the statistical confidence on the damage identification results is important in order to be able to state that the observed changes are statistically significant, and therefore most likely not the result of random variability in

the measurements. Previously, very little has been published on the application of statistical confidence to damage identification. The exception is the work of Mazurek [11], who used the measured confidence intervals on the damage indicator as an inverse-weighting criteria for damage identification.

This study is motivated by previous damage identification analyses of the I-40 data sets that have raised questions about false positive indications due to test-to-test random variations rather than statistically significant changes caused by the actual damage. For example, Farrar and Jauregui [6] applied 5 different damage identification techniques to the I-40 data sets, and 3 of them successfully identified the damage at the lowest level damage case. However, when they attempted to simulate the lowest level of damage from the I-40 tests using a finite element model, they were unable to get the same level of conclusive damage indication as from the experimental result. Of course, this result could be an effect of the inability to properly model the dynamic effects of the damage case, but regardless it raises the question of whether the observed indicators in this lowest level of damage are just a result of random variations. In another study, Alvin [7] showed that the three lowest level damage cases from the I-40 tests did not differ significantly from his FEM than the undamaged case did. Thus, he concluded that the three lower level damage cases did not produce significant enough changes in the modal properties to enable detection of the damage.

The focus of this study is to explore the issue of the statistical significance of the changes in the modal parameters and damage indicators between the various damage cases in the I-40 experiments. The emphasis is on very simple ways to determine the statistical significance of the changes from case to case. The approach demonstrated in this paper uses Monte Carlo analysis to compute statistical confidence intervals on the mode shape components and damage indication parameters (in this case modal flexibility components). Then a statistical dif-

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ference test is employed to associate a confidence level with the statement that the means between the two sample sets are significantly different. This statistical difference test is applied to the individual flexibility components to determine if the flexibility change at that particular degree of freedom (DOF) should be considered to be statistically significant or not. These difference statistics indicate the relative significance of the changes in a particular parameter relative to changes in other parameters.

It should be noted, however, that the results of this analysis can only discriminate whether the observed changes are statistically significant, not whether the source of the observed changes is structural damage. For example, a systematic error in the testing procedure or a change in the structural boundary conditions can both lead to a statistically significant change in measured structural flexibility, which the analyst may falsely conclude is the result of structural damage. The exception is when a change of larger significance is isolated at one particular DOF or set of spatially close DOF. Discriminating between changes resulting from damage and changes resulting from other systematic errors or unknown effects is beyond the scope of this research. The only statement that can be made from this result is whether or not the observed changes at each DOF are statistically significant.

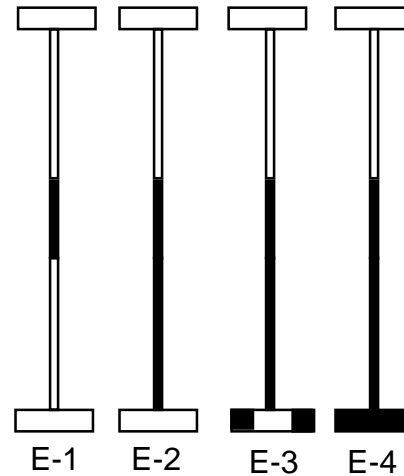
The results presented in this paper indicate that the changes in both the mode shapes and flexibility components are statistically significant for the first three damage cases, but that there is no significant indication of localized flexibility change (and thus damage) until the fourth damage case. The paper begins with a summary of the I-40 experiment and damage cases. Next, an outline of the theory used to perform the modal identification, the damage identification, and the statistical analysis is presented. Following that, the statistical confidence intervals of the modal parameters are presented, as well as the deterministic damage identification results and the values indicating the statistical significance of the damage identification results.

## THE I-40 BRIDGE DAMAGE IDENTIFICATION EXPERIMENT

The research described in this paper uses data from a series of modal tests of a section of a highway bridge. The bridge was located along Interstate Highway 40 across the Rio Grande River in Albuquerque, New Mexico. A series of modal tests was performed on this bridge after it had been closed to traffic prior to demolition in 1993. The bridge was constructed of a concrete deck approximately 13.3 m wide and 17.8 cm thick, supported by two steel plate girders, each 3.05m high, and three steel stringers. The section of the bridge that was instrumented for this series of modal tests consisted of three spans with a combined length of about 130 m. The instrumentation consisted of 13 accelerometers mounted to each of the two plate girders along the length of the three spans, for a total of 26 response measurements. The excitation system consisted of a 9863 kg reaction mass supported by three air springs moved by a 9.79

kN hydraulic actuator. The actuator system was placed on the deck directly over one of the plate girders in the middle of the span closest to the abutment. Full details of the modal testing of this bridge can be found in Farrar, et al. [2].

The damage that was introduced was intended to simulate fatigue cracking that has been observed in plate-girder bridges. Four levels of damage were introduced by making various torch cuts in the web and flange of the girder, as shown in Figure 1. The first level of damage, designated E-1, consisted of a



**Figure 1. The Four Levels of Damage Induced by Cutting the I-40 Plate Girder**

61-cm-long (2 ft.), 10-mm-wide (3/8-in) cut through the web centered at mid-height of the web. Next, this cut was continued to the bottom of the web to produce a second level of damage designated E-2. For the third level of damage, E-3, the flange was then cut halfway in from either side directly below the cut in the web. Finally, the flange was cut completely through for damage case E-4 leaving the top 1.22m of the web and the top flange to carry the load at this location. After this damage case this span of the bridge was sagging about 2 cm at the damage location. This damage location is at DOF 20 in the modal data sets.

## THEORETICAL DEVELOPMENT

The analysis of damage identification considering the statistical significance of the damage indicators requires the determination of statistical confidence intervals. These intervals are determined by first computing confidence intervals on the measured modal data. Next, the statistics are propagated (via Monte Carlo simulations) to the identified modal parameters and then to the damage indicators.

This section of the paper contains an outline of the theory and procedures used in this research. The sections on modal pa-

parameter identification, random error analysis on modal data, Monte Carlo simulation, and damage identification using measured flexibility are abbreviated as this information is presented thoroughly in other published sources.

### Modal Parameter Identification

The method used for the results stated in this paper is the Rational Polynomial curve fit, as presented originally by Richardson and Formenti [12], although in practice any modal parameter identification algorithm could be used. The Rational Polynomial curve fit is based on the expression of the frequency response function (FRF) between a given input-output pair as a ratio of two polynomials. This expression can be written as

$$H_{xy}(\omega) = \frac{\sum_{k=0}^m a_k(j\omega)^k}{\sum_{k=0}^n b_k(j\omega)^k} \quad (1)$$

$$n = 2p$$

$$m = 2p - 1 + r$$

where  $p$  is the number of modes and  $r$  is the number of residual terms. The residual terms account for the influence of modes which are located outside the measurement bandwidth. The rational polynomial form of Eq. (1) can be converted to pole-residue form using standard techniques and from there the modal parameters can be extracted.

### Estimation of Random Error in Measured Modal Data

The first step in this analysis is the determination of random errors on measured FRF data in the form of statistical confidence intervals. Measured data contain errors caused by many sources that result in measured spectral function estimates that are not equal to the actual spectral functions of the structure. The errors that are present in the measured modal data can be divided into two basic categories, as described in Bendat and Piersol [18] and Allemang [13]: bias (systematic) errors and random errors. Bias errors cause the mean of the function estimate not to converge to the actual value of the function as more averages are taken. The random errors on these data are quantified in the form of a standard deviation associated with the mean of the FRF estimate magnitude and phase at each frequency line for each measurement DOF. The details of this computation are presented by Doebling and Farrar [14].

After the mean and the standard deviation of the estimate have been computed, they are usually expressed in association with a statistical confidence level that is defined by the assumed distribution of the estimate. For a normal distribution, plus and minus one standard deviation is the confidence interval for a 68% confidence level and plus or minus two standard deviations

is the confidence interval for (approximately) a 95% confidence level. Thus, given the mean  $\mu$  and the standard deviation  $\sigma$  of an estimate, and assuming that the distribution is Gaussian, there is a 68% probability that the next value measured,  $v$ , will fall into the range

$$\mu - \sigma < v < \mu + \sigma \quad (2)$$

Likewise, there is a 95% probability that the next value measured,  $v$ , will fall into the range

$$\mu - 2\sigma < v < \mu + 2\sigma \quad (3)$$

### Monte Carlo Simulation And Application To Modal Parameter Identification

The next step in the analysis is the estimation of confidence intervals on the measured modal parameter, e.g. the modal frequencies and mode shape components. One approach that can be used to characterize errors on identified modal parameters given the distribution of the errors on the data is known as Monte Carlo simulation, as described in Press, et al. [15]. Monte Carlo simulation is a procedure whereby noisy data sets are repeatedly “simulated” using the assumed statistical distribution of error on the data. The identification procedure is then applied to each of the simulated data sets, producing a set of identified modal parameters. After a sufficient number of simulations, the distribution of the resulting set of identified modal parameters is assumed to be representative of the distribution of the true modal parameters plus the effects of the measurement errors. A complete description of the procedure used to compute the confidence intervals in the modal parameters for this research is presented by Doebling and Farrar [14].

### Damage Identification Using Modal Flexibility Change

The vibration-based damage identification method used in this research idealizes the change in the dynamically measured flexibility matrix as an estimate of the change in the static behavior of the structure. Because the flexibility matrix is defined as the inverse of the static stiffness matrix, the flexibility matrix  $[G]$  relates the applied static force  $\{u\}$  and resulting structural displacement  $\{F\}$  as

$$\{u\} = [G]\{F\} \quad (4)$$

Thus, each column of the flexibility matrix represents the displacement pattern of the structure associated with a unit force applied at the associated DOF.

The measured flexibility matrix is estimated from the mass-normalized measured mode shapes  $[\Phi]$  and squared modal frequencies  $[\Lambda]$  as

$$[G] \approx [\Phi][\Lambda]^{-1}[\Phi]^T \quad (5)$$

The formulation of the flexibility matrix in Eq. (5) is approximate due to the fact that only the first few modes of the structure (typically the lowest-frequency modes) are measured. The synthesis of the complete static flexibility matrix would require the measurement of all of the mode shapes and frequencies.

For this particular study, the difference in the diagonal values of the flexibility matrices before and after damage is used as an indicator of damage, in a manner similar to that presented by Robinson, et al. [16]. The change in the diagonal components represents the change in local flexibility at a point; i.e. the change in the point displacement observed by applying a unit load at that point. This technique is sometimes referred to as the “multi-point flexibility change” technique.

### Confidence Intervals On Flexibility Damage Indicator

A Monte Carlo procedure similar to that used to estimate the confidence intervals on the identified modal parameters is used to estimate the confidence intervals on the values of the flexibility change used as the damage indicator. Specifically, the confidence intervals on the mode shapes and modal frequencies are used to define a normal probability distribution on each component used to compute the flexibility. Then uncertain values of the modal parameters are computed repeatedly and the diagonal values of the flexibility matrix are computed each time for each damage case. After a sufficient number of runs (determined using convergence of the mean and standard deviations as well as the skewness and kurtosis of the distribution, as discussed in Ref. [14]), a statistical distribution is formed on the flexibility change parameters and a mean and confidence intervals are computed. Thus, the damage indicator (in this case the diagonal values of the flexibility change) now has statistical confidence intervals associated with it.

### Tests of Significant Difference Between Sample Means

To assess whether the observed change between the undamaged and damaged flexibility coefficients is statistically significant, a statistical test known as the t-test is used. As described by Freund [17], given two population samples of size  $n_1$  and  $n_2$  with sample means  $\bar{x}_1$  and  $\bar{x}_2$  and sample standard deviations  $S_1$  and  $S_2$ , a test statistic  $z$  can be defined as

$$z = \frac{\bar{x}_1 - \bar{x}_2 - \delta}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \quad (6)$$

to describe the hypothesis  $\bar{x}_1 - \bar{x}_2 = \delta$ , i.e. that the two sample means differ by an arbitrary value  $\delta$ . This formulation assumes that  $n_1$  and  $n_2$  are large enough to invoke the central limit theorem and thus assume normal distributions in the populations.

Using this statistical approach, the objective is to test the hypothesis that one sample mean is statistically larger than the other, written as

$$|\bar{x}_1 - \bar{x}_2| > 0 \quad (7)$$

Solving for  $z$  in Eq. (6) with  $\delta = 0$ , it can then be stated that Eq. (7) is true to approximately the 95% confidence level if  $|z| \geq 2$ . Also, it can be stated that larger values of  $z$  indicate more confidence that the difference in the sample means is statistically significant. We can thus define parameter changes that have larger values of  $z$  as being “more significant changes”.

## EXPERIMENTAL RESULTS, STATISTICAL ANALYSIS, AND DISCUSSION

The most basic sets of results from the identification of the modal parameters using the Monte Carlo estimation technique describe above are the confidence intervals on the identified modal frequencies, modal damping ratios, and mode shape components for each damage case. A comparison of the first modal frequency for each damage case with  $2\sigma$  confidence intervals is shown in Figure 2. As shown in this figure, the  $2\sigma$  confidence intervals are so small relative to the change from case to case that the  $-2\sigma$  and  $+2\sigma$  markers are barely distinguishable from each other. A close up of Figure 2 showing only the undamaged case and the first two damage cases is shown in Figure 3. In this figure, the confidence intervals are distin-

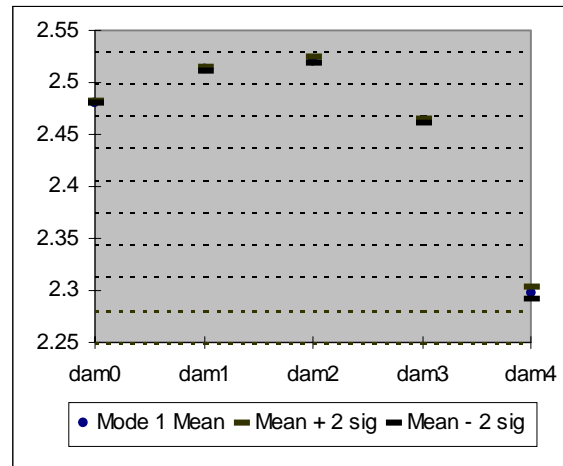
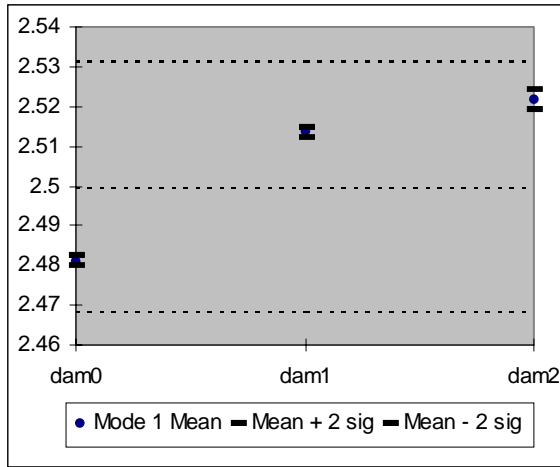


Figure 2. First Modal Frequency for each Damage Case

guishable, but it is still obvious that the change from case to case is much greater than the confidence interval for each case. These results indicate that the changes in modal frequency from one damage case to the next tend to be statistically significant



**Figure 3. First Modal Frequency for First Two Damage Cases**

with respect to the level of uncertainty arising from experimental repeatability.

A comparison of the first mode shape (plotted against DOF number) for the undamaged and first damage cases (including  $2\sigma$  confidence intervals) is shown in Figure 4. This figure demonstrates that there is a slight shift in this mode shape after the first damage state is applied. The confidence intervals at some degrees of freedom clearly overlap, while the confidence intervals at other degrees of freedom do not overlap. In contrast to Figure 4, a comparison of the first mode shape for the undamaged and fourth damage cases is shown in Figure 5. This figure demonstrates that there is a major shift in this mode shape after the fourth damage state is applied. Thus it is clear that there is a significant change in some components of mode shape 1 for damage case 4, but the result is not as conclusive for damage case 1.

A similar comparison of the third mode shape for the undamaged and fourth damage cases is shown in Figure 6. This figure demonstrates that there is a very slight shift in this mode shape even after the fourth damage state is applied. The confidence intervals at almost all degrees of freedom clearly overlap. Upon close inspection of this mode shape, it is apparent that it has a node at DOF 20, which is the location where the damage is applied. Thus it is logical that this mode is insensitive to damage at DOF 20 for all four damage cases.

The previous three comparisons give an idea of the statistical significance of the changes of these mode shapes as a result of damage, but in a purely qualitative manner. To compare the relative level of statistical significance of the changes in these mode shapes after damage in a more quantitative manner, the z-statistics for  $\delta = 0$  were computed using Eq. (6). The values of these z-statistics are shown in Figure 7. The three lines represent the z-statistic values of the differences between each of the mode shape pairs in Figure 4, Figure 5, and Figure 6. As discussed in the theory section, the z-statistics give a mea-

sure of statistical significance of the change of each mode shape component. It is clear from this plot that the only mode shape component that has a significantly larger change than the other components is at DOF 20 for the comparison of the undamaged first mode to the first mode after damage case 4. It can be seen in this plot that the change of mode 1 from undamaged to damage case 1 and the change of mode 3 from undamaged to damage case 4 are of about equal significance. Noting that mode three is almost totally insensitive to the damage, it is logical to assume that the changes in mode shape 3 (as well as mode shape 1 for damage case 1) do not result from damage, but rather from some other change in the experimental conditions.

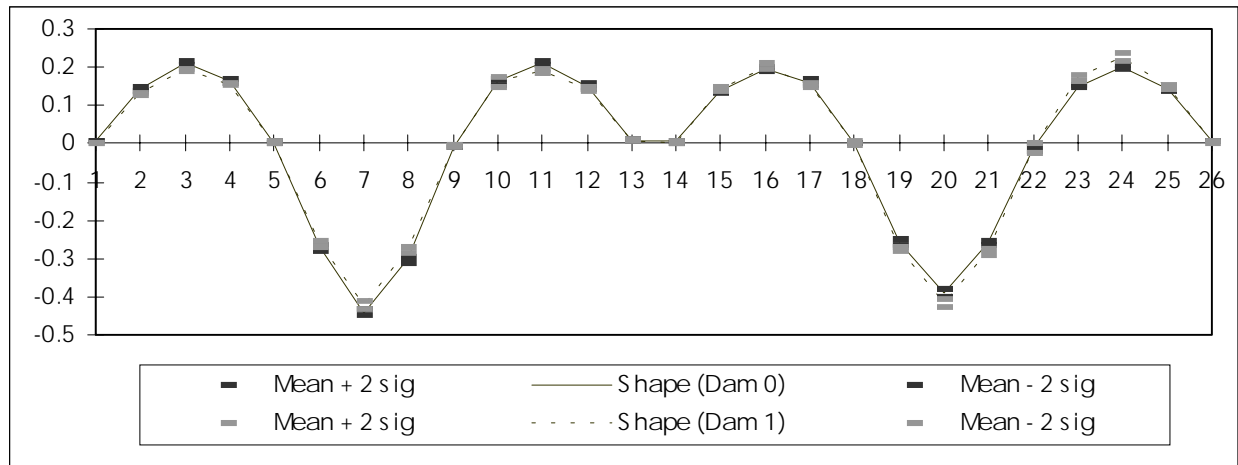
Another comparison of z-statistics, Figure 8, shows the statistical significance of the changes in mode shape 1 for each damage case. This comparison would seem to support the assertion that the fourth damage case is the only one that demonstrates significant change in its modal properties as a result of damage. However, the real measure of the statistical significance of the changes must be made with respect to the damage indicator function and not with respect to the raw modal properties. Such a comparison follows in the next section.

### Damage Identification using Measured Flexibility Change

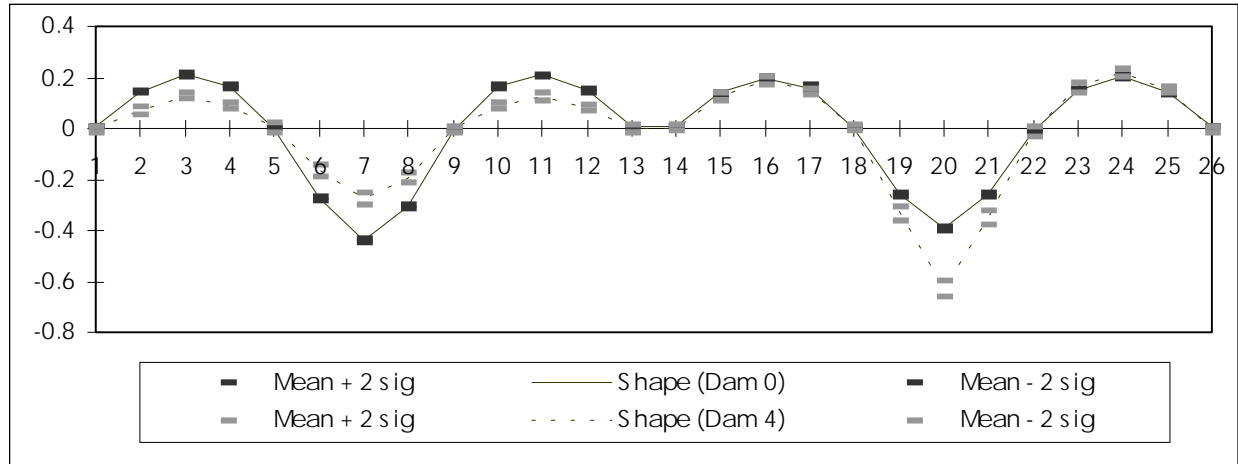
The change in the measured flexibility parameters is used as an indicator of structural damage for the I-40 test results. The changes in the point-flexibility (diagonal values of the flexibility matrix) are shown for each damage case in Figure 9. According to these computations, the largest change in flexibility for each of the damage cases occurs at DOF 20, which is the known location of the damage. The second largest change is at DOF 7, which is the symmetric location across the span from the damage location. However, as mentioned in the introduction, other researchers have found reason to believe that this fact is mere coincidence and that damage is not actually being detected until the final damage case.

To study the statistical significance of the flexibility changes presented in Figure 9, it is necessary to compute statistical confidence intervals on the measured flexibility from the undamaged case and each damage case. The estimation of these intervals is accomplished by means of a Monte Carlo analysis, as explained earlier in the paper.

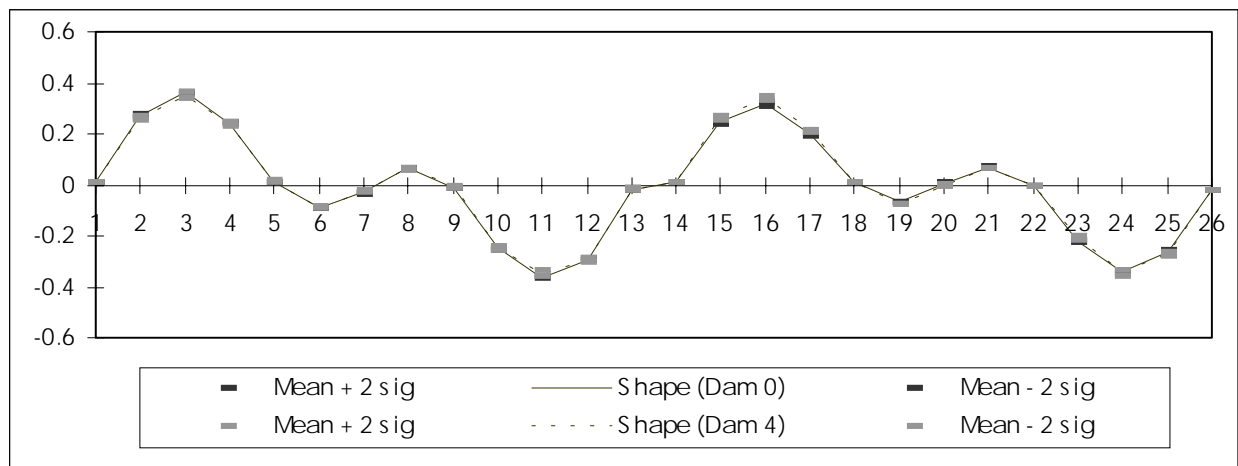
The z-statistics of Eq. (6) with  $\delta = 0$  are again used, this time to determine the relative significance of the changes in the measured flexibility components. A chart showing the z-statistic values for the changes in the flexibility components is presented in Figure 10. Upon examination of these values, the first observation that is apparent is that each damage case demonstrates a significant change from the undamaged case, as proven by the values of z which are much larger than 2 (in fact, are on the order of 10 and 20). However, this result alone only demonstrates that the changes in the structural flexibility are statistically significant with respect to the measurement repeat-



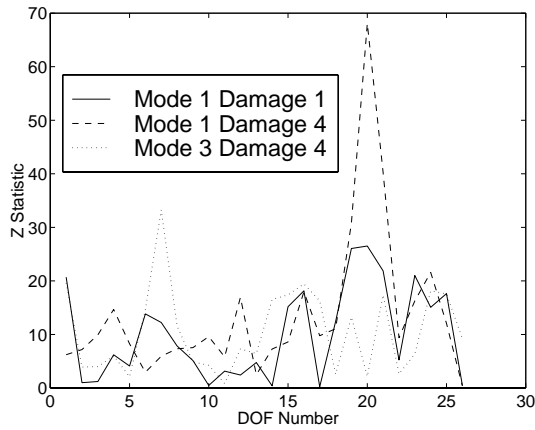
**Figure 4. First Mode Shape for Undamaged Case and First Damage Case**



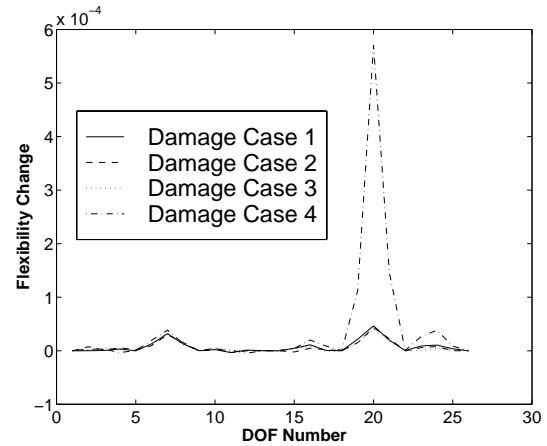
**Figure 5. First Mode Shape for Undamaged Case and Fourth Damage Case**



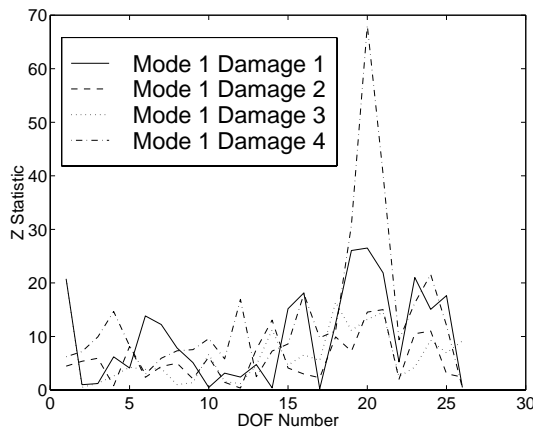
**Figure 6. Third Mode Shape for Undamaged Case and Fourth Damage Case**



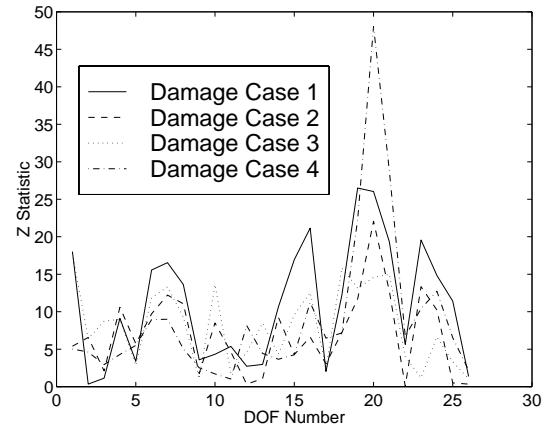
**Figure 7. Comparison of Z-Statistics for Mode Shape Change Comparisons**



**Figure 9. Flexibility Change for All 4 Damage Cases**



**Figure 8. Comparison of Z-Statistics for Mode Shape 1 for Each Damage Case vs. Undamaged**



**Figure 10. Z-statistic values for Flexibility Change for All 4 Damage Cases**

ability, not that the identification of the damage location is statistically significant.

The second observation that is apparent from examination of Figure 10 is that until damage case 4, there is not a single particular flexibility component that is noticeably larger than the others, implying that there is no localized damage. To illustrate this point, consider damage case 1. If the changes in flexibility for damage case 1 were caused by localized damage, then one would expect to see a more significant change at one particular flexibility component. The changes in flexibility of approximately equal significance at several locations on the bridge indicate that variabilities other than localized damage are present in this data set. Thus, it can be stated that all four of the damage cases in the I-40 experiment show statistically significant changes in measured flexibility. However, it cannot be stated that these changes conclusively localize the damage until the fourth and final damage case.

## CONCLUSIONS

The statistical significance of vibration-based damage identification parameters was studied via application to the data from tests performed on the Interstate 40 highway bridge in Albuquerque, New Mexico. A test of statistical significance was applied to the mean and confidence interval estimates of the modal properties and the corresponding indicators of damage in the form of the change in the measured flexibility matrix. Previously presented deterministic results indicate that damage is detectable in all of the damage cases from these data sets. The results of this study indicate that the changes in both the modal properties and the damage indicators are statistically significant for all of the damage cases. However, these changes are distributed spatially for the first three damage cases and do not localize the damage until the fourth and final damage case. Perhaps an approach to consider for future studies of this type is the gen-

eration of a “damage noise floor” by computing damage indicators between multiple undamaged data sets. This approach would produce a baseline value for the damage indicator at each DOF that could be expected as a result of test-to-test variations. The damage indicators computed using subsequent data sets could then be compared to the damage noise floor to quantify the statistical significance of the structural damage.

## ACKNOWLEDGMENTS

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